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# ON GENERIC FINITENESS OF PLURICANONICAL MAPS OF 3-FOLDS OF GENERAL TYPE

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## Abstract

For a complex projective 3-fold of general type, if $\dim\phi_k(X) \geq 2$, it is proved in this paper that the $m$-canonical map is generically finite when $m \geq 2k+2$ and $(9k+9)$-canonical map is birational. This serves as a complementary result to that of Kollár and Tankeev. Our method can work on any higher dimensional variety of general type which admits a minimal model. For a 3-fold of general type with canonical index $r$, an improved version of Hanamura's theorem on the birationality of pluricanonical maps is supplied. We also give a function $m(r)$ such that $\phi_m$ is a map of generically finite when $m \geq m(r)$. Finally we present our further results on $\phi_3$, $\phi_4$ and $\phi_5$ for nonsingular minimal 3-folds of general type. The effective $\mathbb{Q}$-divisor trick through the Kawamata-Viehweg vanishing is employed in our arguments.

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## Introduction

Let $X$ be a complex nonsingular projective threefold of general type. We denote by $\phi_m$ pluricanonical map of $X$, which is just the rational map associated with the complete linear system $|mK_X|$, where $m$ is a positive integer. The behavior of $\phi_m$ plays a very important role to the classification theory. Many authors such as T. Ando, X. Benveniste, M. Hanamura, Y. Kawamata, K. Kodaira, J. Kollár, S. Lee, T. Luo, K. Matuski, P. Wilson as far as we know, have studied in this area. In this paper, we mainly consider the following problem:

Problem. *Let $X$ be a nonsingular projective variety of general type of dimension $d$. For which value $m(d)$, does $\phi_m$ define a generically finite map onto its image for $m \geq m(d)$?*

For surfaces, it is well-known that we can take $m(2) = 3$. For nonsingular minimal threefolds of general type, an updated Benveniste-Matsuki-Wilson theorem in [2] shows that $\phi_m$ is birational onto its image for $m \geq 6$. Furthermore, S. Lee [14] proved that $\phi_m$ is actually a birational morphism under the same situation by Ein-Lazarsfeld type of arguments. For a nonsingular threefold with canonical index $r \geq 2$, M. Hanamura [7] gave a function $m(3,r)$ such that $\phi_m$ is birational for $m \geq m(3,r)$. J. Kollár [12] studied irregular threefolds. In [15], T. Luo studied regular threefolds partially and, in [16], he gave a function $m(3,\rho)$, in terms of the Picard number $\rho$, such that $\phi_m$ is birational for $m \geq m(3,\rho)$ for arbitrary threefolds. For varieties of dimension $\geq 4$, T. Ando [1] studied the nonsingular minimal models. Now, we list two known theorems.

Theorem A. *(Kollár-Tankeev [12]) Let $X$ be a $d$-dimensional nonsingular projective variety of general type, and $|L|$ be a linear system which gives a rational map onto an $(d-1)$-dimensional variety. Then $|K_X + dL|$ gives a generically finite map.*

Theorem B. *(Kollár [12]) Let $X$ be a 3-fold of general type. If $P_k(X) \geq 2$, where $P_k(X)$ is the plurigenus of $X$, then the $(7k+3)$-canonical map is generically finite and the $(11k+5)$-canonical map is birational.*

By the $\mathbb{Q}$-divisor method and Kawamata-Viehweg vanishing theorem, we obtain the following results.

Theorem 1. *Let $X$ be a $d$-dimensional nonsingular projective variety of general type, $d \geq 3$, and suppose $X$ has a minimal model. If the $|L_i|$, $i = 1, \cdots, d-1$, are linear systems such that $\dim\Phi_{|L_i|}(X) \geq i$ for all $i$. Then $|mK_X + L_1 + \cdots + L_{d-1}|$ gives a generically finite map for $m \geq 2$.*

We have instant applications which seem to be supplementary to both Theorem A and Theorem B.

Corollary 1. *Let $X$ be a nonsingular projective 3-fold of general type. If $P_k(X) \geq 2$ and $\dim\phi_{k+a_0}(X) \geq 2$, then $\phi_m$ is generically finite for $m \geq 2k + a_0 + 2$ and, therefore, $|(9k + 4a_0 + 9)K_X|$ gives a birational map. In particular, if $\dim\phi_k(X) \geq 2$, then $\phi_m$ is generically finite for $m \geq 2k+2$ and $|(9k+9)K_X|$ gives a birational map.*

Corollary 2. *Let $X$ be a nonsingular projective variety of general type with dimension $d \geq 4$. Suppose $X$ has a minimal model and $\dim\phi_{r_i}(X) \geq i$, $i = 1, \cdots, d-1$, then $\phi_m$ is generically finite for $m \geq \sum_{i=1}^{d-1} r_i + 2$ and $|(5\sum_{i=1}^{d-2} r_i + 4r_{d-1} + 9)K_X|$ gives a birational* *map. In particular, if* $\dim\phi_k(X) \geq d-1$*, then* $\phi_m$ *is generically finite for* $m \geq k(d-1)+2$ *and* $|(k(5d-6)+9)K_X|$ *gives a birational map.*

Theorem 2. *Let* $X$ *be a nonsingular projective threefold of general type which has a minimal model of canonical index* $r$*. Then there is a function* $m(r)$ *such that the* $m$*-canonical map of* $X$ *is generically finite for* $m \geq m(r)$*, where* $m(r)$ *is as follows:*

$m(1)=5$*;* $m(r)=2r+7$ *for* $2 \leq r \leq 5$*;* $m(r)=2r+6$ *for* $r \geq 6$*.*

Theorem 3. *Let* $X$ *be a nonsingular projective threefold with nef and big* $K_X$*, then*

*(1)* $\phi_5$ *is birational except when* $K_X^3 = 2$ *and* $0 \leq p_g(X) \leq 2$*;* $\phi_5$ *is generically finite of degree* $\leq 8$*, furthermore, if* $\deg(\phi_5) > 2$*, then* $\chi(\mathcal{O}_X) = -1$ *and* $p_g(X) = 0, 1$*.*

*(2) If* $K_X^3 > 2$ *and* $\dim\phi_1(X) = 3$*, then* $\phi_4$ *is birational onto its image; if either* $K_X^3 > 2$ *or* $p_g(X) \geq 3$ *or* $\chi(\mathcal{O}_X) \neq -1$*, then* $\phi_4$ *is a generically finite map.*

*(3) If* $p_g(X) \geq 39$*, then* $\phi_3$ *is a generically finite map.*

In section 1, we give some basic definitions and notations. Section 2 is devoted to the proof of Theorem 1. The final two sections mainly deal with the proof of Theorem 2. In section 4, we will present our results on generic finiteness of $\phi_m$ for threefolds of index 1 for $m \leq 5$. Unfortunately, it still remains incomplete.

## §1. Preliminaries

Let $X$ be a normal projective variety of dimension $d$ over $\mathbb{C}$. We denote by $Z_{d-1}(X)$ the group of Weil divisors and $\mathrm{Div}(X)$ the group of Cartier divisors on $X$. An element $D \in Z_{d-1}(X) \otimes \mathbb{Q}$ is called a $\mathbb{Q}$-divisor. An element of $\mathrm{Div}(X) \otimes \mathbb{Q}$ is called a $\mathbb{Q}$-Cartier divisor. Two $\mathbb{Q}$-divisors $D$, $D'$ are said to be $\mathbb{Q}$-linearly equivalent, denoted by $D \sim_{\mathbb{Q}} D'$, if there is a positive integer $e$ such that $eD$ and $eD'$ are linearly equivalent in the ordinary sense. A $\mathbb{Q}$-Cartier divisor is said to be nef if some multiple of it is a nef Cartier divisor. A divisor $D$ is called big if the Iitaka dimension $k(X,D)=d$.

Let $D = \sum a_i D_i \in Z_{d-1}(X) \otimes \mathbb{Q}$, where the $a_i$ are rational numbers and the $D_i$ are mutually distinct prime divisors. We define the round up of $D$ as

$$\lceil D \rceil := \sum \lceil a_i \rceil D_i,$$

where $\lceil a_i \rceil = -[-a_i]$ and $[-a_i]$ the integral part of $-a_i$.

Definition 1.1. The canonical divisor $K_X$ is an element of $Z_{d-1}(X)$ such that the reflexive sheaf $\mathcal{O}_X(K_X) = i_*\mathcal{O}_{X^0}(K_{X^0})$, where $X^0$ is the nonsingular locus of $X$ and $i : X^0 \hookrightarrow X$ is the natural inclusion. We also use the following notations:

$$\omega_X^{[r]} := \mathcal{O}_X(rK_X) \text{ for any positive integer } r.$$

Definition 1.2. A normal variety $X$ is said to have only canonical (resp. terminal) singularities if the following two conditions are satisfied:

(1) $K_X$ is $\mathbb{Q}$-Cartier;

(2) there exists a resolution of singularities $f : Y \longrightarrow X$ such that $K_Y = f^*(K_X) + \sum a_i E_i$ for $a_i \in \mathbb{Q}$ with $a_i \geq 0$ (resp. $a_i > 0$) for all $i$, where the $E_i$ vary among all the prime divisors which are exceptional with respect to $f$.

If $X$ has only canonical singularities and $K_X$ is nef, then we say that $X$ is minimal. We use [11] as a good handbook for the rest of other notations and terms. In this paper, for $d \geq 4$, we say that $X$ has a minimal model if the minimal model conjecture is true with respect to the successive contractions from $X$, though it does not seem standard to put the concept in this way.

Remark 1.1. Under the above definitions, if $X$ has a minimal model $X_1$, then we can take a common birational nonsingular modification $X'$ with $g : X' \longrightarrow X$ and $h : X' \longrightarrow X_1$ such that $h = f \circ g$, where $f : X - \cdots \to X_1$ is the contraction map, and that $h^*(K_{X_1}) \leq g^*(K_X)$ as $\mathbb{Q}$-divisors on $X'$.

We will use the Kawamata-Viehweg vanishing theorem in the following form:

Theorem 1.1. *Let $X$ be a nonsingular projective variety, and $D$ be a $\mathbb{Q}$-divisor on $X$ and the fractional part of $D$ has support with only normal crossings. Assume that $D$ is nef and big, then $H^i(X, \mathcal{O}_X(K_X + \lceil D \rceil)) = 0$ for any $i > 0$.*

## §2. The proof of theorem 1

Lemma 2.1. *Let $X$ be a nonsingular projective variety of dimension $d$. Suppose we have a nef and big $\mathbb{Q}$-divisor $L$ and a smooth irreducible divisor $S$ on $X$, then $\lceil L \rceil|_S \geq \lceil L|_S \rceil$ as $\mathbb{Q}$-divisors on $S$, where we take $L|_S$ as a $\mathbb{Q}$-divisor on $S$.*

*Proof.* We can write $L = D + \sum a_i E_i$, where $D$ is a divisor, $0 < a_i < 1$ and the $E_i$ are effective distinct prime divisors. Then

$$\lceil L \rceil|_S = D|_S + \sum E_i|_S = D|_S + \sum b_{ij} G_{ij},\ \ 0 < b_{ij} \in \mathbb{Z}^+;$$
$$\lceil L|_S \rceil = D|_S + \sum \lceil a_i b_{ij} \rceil G_{ij}.$$

The lemma is proved. □

Lemma 2.2. *Let $f : X' \longrightarrow X$ be a birational morphism between nonsingular varieties, $L$ a $\mathbb{Q}$-divisor on $X$, then we have $f^*(\lceil L \rceil) \geq \lceil f^*(L) \rceil$ as $\mathbb{Q}$-divisors.*

*Proof.* The proof is similar to that in the previous lemma. □

Theorem 2.1. *Let $X$ be a nonsingular projective variety of general type of dimension $d \geq 2$. Suppose that $L_0$ is a nef and big $\mathbb{Q}$-divisor on $X$ and that the $|L_i|$ are linear systems such that $\dim \Phi_{|L_i|}(X) \geq i$ for $i = 1, \cdots, d-1$, then $|K_X + \lceil L_0 \rceil + L_1 + \cdots + L_{d-1}|$ gives a generically finite map onto its image.*

*Proof.* We want to formulate our proof by induction on the dimension $d$ of $X$. First of all, we can take a birational modification $f : X' \longrightarrow X$ such that the fractional part of $f^*(L_0)$ has support with only normal crossings and that the $|f^*(L_i)|$ are base point free for $i = 1, \cdots, d-1$. From Lemma 2.2, we see that

$$\begin{aligned} &|K_{X'} + \lceil f^*(L_0) \rceil + f^*(L_1) + \cdots + f^*(L_{d-1})| \\ &\subset |K_{X'} + f^*(\lceil L_0 \rceil) + f^*(L_1) + \cdots + f^*(L_{d-1})|. \end{aligned}$$

Therefore replacing $X$ by $X'$, we can suppose that the fractional part of $L_0$ has support with only normal crossings and the $|L_i|$ are base point free for $i > 0$.

If $d=3$, we set $L_i \sim_{\rm lin} S_i + Z_i$, where $S_i$ is the moving part and $Z_i$ is the fixed part for $i=1,2$. Obviously, we have

$$|K_X + \lceil L_0 \rceil + S_1 + S_2| \subset |K_X + \lceil L_0 \rceil + L_1 + L_2|.$$

When $\dim \Phi_{|L_1|}(X) = 1$, we have $S_1 \sim_{\rm num} a_1 F$, where $F$ is a nonsingular projective surface of general type. Let $\Phi_{|L_1|} := s_1 \circ h_1 : X \stackrel{h_1}{\to} W_1 \stackrel{s_1}{\to} W_1'$ be the Stein factorization and take a generic decomposition of $S_1$ into $\sum F_i$, from the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_X(K_X + \lceil L_0 \rceil + S_2) \longrightarrow \mathcal{O}_X(K_X + \lceil L_0 \rceil + S_2 + \sum_{i=1}^{a_1} F_i) \\ \longrightarrow \oplus_{i=1}^{a_1} \mathcal{O}_{F_i}(K_{F_i} + \lceil L_0 \rceil|_{F_i} + S_2|_{F_i}) \longrightarrow 0, \end{aligned}$$

we obtain by the Kawamata-Viehweg vanishing theorem that

$$\Phi_{|K_X + \lceil L_0 \rceil + S_2 + S_1|}|_{F_i} = \Phi_{|K_{F_i} + \lceil L_0 \rceil|_{F_i} + S_2|_{F_i}|}.$$

When $\dim \Phi_{|L_1|}(X) \geq 2$, then a general member of $|S_1|$ is an irreducible nonsingular projective surface of general type by Bertini's theorem. We simply denote $S_1$ as such a general surface. Similarly, we can also see that

$$\Phi_{|K_X + \lceil L_0 \rceil + S_2 + S_1|}|_{S_1} = \Phi_{|K_{S_1} + \lceil L_0 \rceil|_{S_1} + S_2|_{S_1}|}.$$

Anyway, we obtain a family of nonsingular projective surface $S$ of general type such that

$$\Phi_{|K_X + \lceil L_0 \rceil + S_2 + S_1|}|_S = \Phi_{|K_S + \lceil L_0 \rceil|_S + L_2'|},$$

where $|L_2'| = |S_2|_S|$ is a divisor on $S$ with $\dim \Phi_{|L_2'|}(S) \geq 1$.

From Lemma 1.1, we see that $\lceil L_0 \rceil|_S \geq \lceil L_0|_S \rceil$ as $\mathbb{Q}$-divisors on $S$. In order to prove the case when $d=3$, we only have to prove that $\Phi_{|K_S + \lceil L_0' \rceil + L_2'|}$ is a generically finite map onto its image, where $L_0' := L_0|_S$ is a nef and big $\mathbb{Q}$-divisor on $S$. As we have just seen that we can suppose $|L_2'|$ be free. Now take a similar argument as above, let $C$ be a general irreducible curve in the moving part $M_2'$ of $|L_2'|$, which forms a family on $S$, we have

$$|K_S + \lceil L_0' \rceil + M_2'||_C \geq |K_C + \lceil L_0' \rceil|_C|.$$

Note that $C$ is a curve of genus $\geq 2$ and $\deg(\lceil L_0' \rceil|_C) \geq 1$, then we see that $h^0(C, \mathcal{O}_C(K_C + \lceil L_0' \rceil|_C)) \geq 2$. Which shows that $|K_S + \lceil L_0' \rceil + L_2'|$ gives a generically finite map and therefore, Theorem 1.1 is true when $d=3$.

If $d > 3$, the induction step is obvious. So we get the theorem. $\square$

From the proof of Theorem 2.1, we actually have the following slightly stronger result.

THEOREM 2.2. *Let $X$ be a nonsingular projective 3-fold of general type, $L_0$ is a nef and big $\mathbb{Q}$-divisor on $X$. Suppose $|L_i|$ are linear systems with $h^0(X, \mathcal{O}_X(L_i)) \geq 2$ for $i=1,2$ and the $|L_i|$ do not determine the same pencils on $X$, then $|K_X + \lceil L_0 \rceil + L_1 + L_2|$ defines a generically finite map onto its image.*

THEOREM 2.3. *Let $X$ be a nonsingular projective variety of general type with dimension $d \geq 3$ and $X$ has a minimal model. Suppose the $|L_i|$ are linear systems such that* $\dim\Phi_{|L_i|}(X) \geq i$ *for $i = 1, \cdots, d-1$, then $|mK_X + L_1 + \cdots + L_{d-1}|$ defines a generically finite map onto its image for $m \geq 2$.*

*Proof.* Let $f : X - \cdots \to X_1$ be successive contraction maps onto a minimal model $X_1$, according to Remark 1.1 and Hironaka's big theorem, we can take a common birational modification $X'$ with $g : X' \longrightarrow X$ and $h : X' \longrightarrow X_1$ such that $h = f \circ g$, $h^*(K_{X_1}) \leq g^*(K_X)$, and the fractional part of $g^*(K_X)$ has support with only normal crossings. We have the relation

$$\begin{aligned}&|K_{X'} + \lceil h^*((m-1)K_{X_1})\rceil + g^*(L_1) + \cdots + g^*(L_{d-1})|\\ &\subset |K_{X'} + (m-1)g^*(K_X) + g^*(L_1) + \cdots + g^*(L_{d-1})|.\end{aligned}$$

From Theorem 2.1, we see that the former system defines a generically finite map, so does the latter one and thus we obtain the theorem. □

THEOREM 2.4. *Let $X$ be a nonsingular projective variety of general type with dimension $d \geq 2$, $L$ a nef and big divisor on $X$. Suppose the $|L_i|$ are systems such that* $\dim\Phi_{|L_i|}(X) \geq i$ *for $i = 1, \cdots, d-2$, then $|K_X + 4L + L_1 + \cdots + L_{d-2}|$ gives a birational map onto its image.*

*Proof.* This is a rewrite of Theorem 2.1 of [4]. One can also refer to [6] for the technique of the proof. □

REMARK 2.1. In Theorem 2.4, when $d = 3$, let $S$ be a general irreducible member of the moving part of $|L_1|$. If, furthermore, $L^2 \cdot S \geq 2$, then $|K_X + 3L + L_1|$ defines a birational map onto its image.

DEFINITION 2.1. Let $X$ be a nonsingular projective variety of general type with dimension $d$. We set

$r_i(X) := \min\{k|\dim\phi_k(X) \geq i\}$, for $i = 1, \cdots, d-1$;

$\tilde{r_1}(X) := \min\{k|\dim\phi_m(X) \geq 1$ for all $m \geq k\}$.

COROLLARY 2.1. *Let $X$ be a nonsingular projective variety of general type with dimension $d$. Then $\phi_m$ is generically finite for $m \geq \sum_{i=1}^{d-1} r_i(X) + 2$. $\phi_m$ is birational for $m \geq 5\sum_{i=2}^{d-2} r_i(X) + 4(r_1(X) + r_{d-1}(X)) + \tilde{r_1}(X) + 9$.*

*Proof.* The first part is obvious according to Theorem 2.3. In order to prove the second part, we take a birational modification $f : X' \longrightarrow X$ according to Hironaka such that $f^*((\sum_{i=1}^{d-1} r_i(X) + 2)K_X)$ is free from base points. Let $L$ be the moving part of it, then $L$ is nef and big. Take $L_0 = L$, $L_i = r_i(X)K_{X'}$ for $i = 2, \cdots, d-2$ and $L_1 = f^*(pK_X)$ with $p = m - 5\sum_{i=2}^{d-2} r_i(X) - 4(r_1(X) + r_{d-1}(X)) - 9$, using Theorem 2.4, we can get the result. □

Both Corollary 1 and Corollary 2 follow from Corollary 2.1.

## §3. THREEFOLDS WITH INDEX $\geq$ 2

In this section, we shall first update Hanamura's theorem. Then we present Theorem 2 as an application of our Theorem 1.

Lemma 3.1. *(Lemma 2.3 of [7]) Let $X$ be a minimal threefold of general type with canonical index $r$. Then we have the plurigenus formula*

$$\begin{aligned} & h^0(X, \omega_X^{[mr+s]}) \\ & = \frac{1}{12}(mr+s)(mr+s-1)(2mr+2s-1)(K_X^3) + am + c_s \end{aligned}$$

*for $0 \leq s < r$, $mr+s \geq 2$, where $a$ is a constant and $c_s$ is a constant only relating to $s$.*

Theorem 3.1. *(Theorem 3.4 of [7]) Let $X$ be a nonsingular threefold of general type with a minimal model of canonical index $r \geq 2$. Then $\phi_m$ is birational for $m \geq m_0(r)$ where $m_0(2) = 13$, $m_0(r) = 4r+4$ for $3 \leq r \leq 5$ and $m_0(r) = 4r+3$ for $r \geq 6$.*

Now we modify Lemma 3.2 of [7] as follows.

Proposition 3.1. *Let $X$ be a minimal threefold of general type with a minimal model of canonical index $r \geq 2$. Then*

*(i) $h^0(mr+s) \geq 3$ in one of the following cases:*

*(i1) $r = 2$ and $m \geq 2$;*

*(i2) $r \geq 3$, $s = 0, 1$ and $m \geq 2$; $r \geq 3$, $s \geq 2$ and $m \geq 1$.*

*(ii) $\dim \phi_{mr+s}(X) \geq 2$ in one of the following cases:*

*(ii1) $r = 2$ and $m \geq 3$;*

*(ii2) $r = 3$ and $m \geq 2$;*

*(ii3) $r = 4, 5$, $0 \leq s \leq 2$ and $m \geq 2$; $r = 4, 5$, $s \geq 3$ and $m \geq 1$;*

*(ii4) $r \geq 6$, $0 \leq s \leq 1$ and $m \geq 2$; $r \geq 6$, $s \geq 2$ and $m \geq 1$.*

*Proof.* The second part is according to Hanamura. We only have to show the first part.

From Lemma 3.1, we can put

$$P(mr+s) = \frac{1}{12}(mr+s)(mr+s-1)(2mr+2s-1)(K_X^3) + am + c_s, \qquad ((1))$$

where $a$ and $c_s$ are constants for $0 \leq s < r$. We consider the right hand of (1) as a polynomial in $m$ and denote it by $P_s(m)$. Let $Q_s(m)$ be the first term of $P_s(m)$, we have

$$P_s(m) = Q_s(m) + am + c_s.$$

We see that, for $m \geq 1$ or $m = 0$ and $s \geq 2$,

$$P_s(m) \geq 0 \qquad ((2)_{s,m})$$

By Kollàr's result that the $\omega_X^{[mr+s]}$ are Cohen-Macaulay, using Grothendieck duality, one can see that, for $m \leq -1$,

$$P_s(m) \leq 0 \qquad ((2)_{s,m})$$

Now we want to estimate both $a$ and $c_s$. For any $r$ and $s$, we have

$$Q_s(1) + a + c_s \geq 0 \qquad ((2)_{s,1})$$

$$-Q_s(-1) + a - c_s \geq 0 \qquad ((2)_{s,-1})$$

Which induces

$$a \geq \frac{1}{2}\{Q_s(-1) - Q_s(1)\}$$
$$= -\frac{1}{12}\{2r^2 + (6s^2 - 6s + 1)\}(rK_X^3) \tag{$(3)_s$}$$

When $r \geq 3$, for $s \geq 2$, we have

$$Q_s(0) + c_s \geq 0 \tag{$(2)_{s,0}$}$$

By $(2)_{s,-1}$ and $(2)_{0,0}$, we get

$$a \geq -Q_s(0) + Q_s(-1)$$
$$= \frac{1}{12}\{-2r^2 + (6s-3)r - (6s^2 - 6s + 1)\}(rK_X^3) \tag{$(4)_s$}$$

Explicitly, we have

$$r \text{ odd}, \ a \geq \frac{1}{12}\{-\frac{1}{2}r^2 + \frac{1}{2}\}(rK_X^3) \tag{$(4)_{(r+1)/2}$}$$
$$r \text{ even}, \ a \geq \frac{1}{12}\{-\frac{1}{2}r^2 - 1\}(rK_X^3) \tag{$(4)_{r/2}$}$$

Now we can calculate the $P(mr+s)$ case by case.

Case 1. $r \geq 3$ and $s \geq 2$.

When $r$ is odd, from $(2)_{s,0}$ and $(4)_{(r+1)/2}$, we have

$$P(mr+s) \geq Q_s(m) - \frac{1}{12}m(\frac{1}{2}r^2 - \frac{1}{2})(rK_X^3) - Q_s(0)$$
$$= \frac{1}{12}\{(mr+s)(mr+s-1)(2mr+2s-1) + m(-\frac{1}{2}r^3 + \frac{1}{2}r)$$
$$- s(s-1)(2s-1)\}(K_X^3)$$

We get $P(mr+s) \geq 7$ for $m \geq 1$.

When $r$ is even, from $(2)_{s,0}$ and $(4)_{r/2}$, we have

$$P(mr+s) \geq Q_s(m) - \frac{1}{12}m(\frac{1}{2}r^2 + 1)(rK_X^3) - Q_s(0)$$
$$= \frac{1}{12}\{2r^2m^3 + (6s-3)rm^2 + (6s^2 - 6s - \frac{1}{2}r^2)m\}(rK_X^3)$$

We get $P(mr+s) \geq 5$ for $m \geq 1$.

Case 2. $s = 1$.

From $(2)_{1,1}$ and $(2)_{1,-1}$, we have

$$P(mr+1) \geq \frac{1}{12}r(m^2-1)(2rm+3)(rK_X^3).$$

We get $P(mr+1) \geq 6$ for $m \geq 2$.

Case 3. $s = 0$.

By $(2)_{0,1}$ and $(2)_{0,-1}$, we have

$$P(mr) \geq \frac{1}{12}r(m^2-1)(2rm-3)(rK_X^3).$$

We get $P(mr) \geq 3$ for $m \geq 2$. Thus we obtain item (i). $\square$

In what follows we can get an improved version of Theorem 3.1 as follows.

Theorem 3.1$'$. *Let $X$ be a nonsingular threefold of general type which has a minimal model of index $r \geq 2$. Then $\phi_m$ is birational onto its image for $m \geq 4r+3$.*

*Proof.* Put $k = m - 3r - 1$ and let $f : X - \cdots \to X_1$ be successive contraction maps onto a minimal model. Take a common modification $X'$ with $g : X' \longrightarrow X$ and $h : X' \longrightarrow X_1$ such that $h = f \circ g$ and $h^*(K_{X_1}) \leq g^*(K_X)$. Now on $X'$, take $L = h^*(rK_{X_1})$ and $L_1 = kK_{X'}$, using Proposition 3.1 and Theorem 2.4 as well as Remark 2.1, we obtain the theorem. □

Theorem 3.2. *Let $X$ be a nonsingular projective threefold of general type which has a minimal model of index $r \geq 2$. Then $\phi_m$ is a generically finite map onto its image for $m \geq m(r)$ where $m(r) = 2r+7$ for $2 \leq r \leq 5$; $m(r) = 2r+6$ for $r \geq 6$.*

*Proof.* Let $f : X - \cdots \to X_1$ be successive contraction maps onto a minimal model $X_1$. Take a common modification $X'$ with $g : X' \longrightarrow X$ and $h : X' \longrightarrow X_1$ such that $h = f \circ g$ and $h^*(K_{X_1}) \leq g^*(K_X)$. If $r = 2$, according to Theorem 3.1$'$, the theorem is true. If $3 \leq r \leq 5$, according to Proposition 3.1, we can take $L_1 = a(r)K_{X'} = (r+2)K_{X'}$ and $L_2 = b(r)K_{X'} = (r+3)K_{X'}$; if $r \geq 6$, take $L_1 = L_2 = a(r)K_{X'} = b(r)K_{X'} = (r+2)K_{X'}$. Also we put $p = m - a(r) - b(r) - 1$ and $L_0 = h^*(pK_{X_1})$. Then we use our Theorem 2.1 to obtain the generic finiteness of $\phi_m$. □

## §4. Threefolds with index 1

For a minimal threefold $X$ of general type with canonical index 1, we can find certain birational modifications $f : X' \longrightarrow X$ according to M. Reid [20] such that $c_2(X') \cdot \Delta = 0$, where $\Delta$ is the ramification divisor of $f$. Then we can get the same plurigenus formula as that for a nonsingular minimal threefold. On the other hand, the Miyaoka-Yau inequality shows that $\chi(\mathcal{O}_X) < 0$. From Ein-Lazarsfeld-Lee [14] and [2], we know that $\phi_m$ is a birational morphism for $m \geq 6$. Actually, $\phi_5$ is a generically finite morphism. Therefore we can take $m(1) = 5$ in Theorem 2.

In this section, we want to present further results on $\phi_3$, $\phi_4$ and $\phi_5$. Though we can work on an arbitrary 3-fold of index 1 in the same way, we prefer to study a nonsingular minimal one.

Theorem 4.1. *([5]) Let $X$ be a nonsingular minimal 3-fold of general type. Then*

*(1) If either $K_X^3 > 2$(Ein-Lazarsfeld-Lee) or $p_g(X) > 2$, then $\phi_5$ is a birational morphism.*

*(2) If $p_g(X) = 2$ and $\phi_5$ is not birational, then $\phi_5$ is generically finite of degree 2 and $q(X) = h^2(\mathcal{O}_X) = 2$ and $|K_X|$ is composed with a rational pencil of Horikawa surface with $(K^2, p_g) = (1, 2)$.*

*(3) If $\dim\phi_2(X) = 1$, then $\phi_5$ is birational.*

Definition 4.1. Let $X$ be a nonsingular projective 3-fold with nef and big canonical divisor $K_X$. Suppose $\dim\phi_i(X) \geq 2$ and set $iK_X \sim_{\text{lin}} M_i + Z_i$, where $M_i$ is the moving part and $Z_i$ the fixed one for any integer $i$. We define $\delta_i(X) := K_X^2 \cdot M_i$.

Proposition 4.1. *Let $X$ be a nonsingular projective 3-fold with nef and big $K_X$. Suppose $|2K_X|$ is not composed of pencils and $K_X^3 > 2$, then $\delta_2(X) \geq 3$.*

*Proof.* We have $\delta_2(X) \geq 2$ by Proposition 2.2 of [2]. Take a birational modification $f : X' \longrightarrow X$ such that $|2f^*(K_X)|$ defines a morphism. Set $2f^*(K_X) \sim_{\text{lin}} M + Z$, where $M$ is the moving part and $Z$ the fixed one. A general member $S \in |M|$ is an irreducible nonsingular projective surface of general type. Denote $L := f^*(K_X)|_S$, if $L^2 = f^*(K_X)^2 \cdot S = \delta_2(X) = 2$, then we have

$$4 = 2f^*(K_X)^2 \cdot S = f^*(K_X) \cdot S^2 + f^*(K_X) \cdot S \cdot Z.$$

Note that $S$ is nef and $S \not\approx 0$, we have $f^*(K_X) \cdot S^2 \geq 1$. Therefore four cases occur as follows:

(i) $f^*(K_X) \cdot S^2 = 4$, $f^*(K_X) \cdot S \cdot Z = 0$;
(ii) $f^*(K_X) \cdot S^2 = 3$, $f^*(K_X) \cdot S \cdot Z = 1$;
(iii) $f^*(K_X) \cdot S^2 = 2$, $f^*(K_X) \cdot S \cdot Z = 2$;
(iv) $f^*(K_X) \cdot S^2 = 1$, $f^*(K_X) \cdot S \cdot Z = 3$.

We also have

$$\begin{aligned} 2K_X^3 = 2f^*(K_X)^3 &= f^*(K_X)^2 \cdot S + f^*(K_X)^2 \cdot Z \\ &= 2 + \frac{1}{2} f^*(K_X) \cdot Z(S+Z) \\ &= 2 + \frac{1}{2} f^*(K_X) \cdot S \cdot Z + \frac{1}{2} f^*(K_X) \cdot Z^2 \end{aligned} \tag{A}$$

Case (i). Noting that $f^*(K_X)$ is nef and big, we see that $mf^*(K_X)$ is linearly equivalent to a nonsingular projective surface of general type according to Kawamata for sufficiently large integer $m$. Then $S|_{mf^*(K_X)}$ is nef and big and, by the Hodge Index Theorem, we have $f^*(K_X) \cdot Z^2 \leq 0$. Thus (A) is false and this case does not occur.

Case (ii). We have $f^*(K_X) \cdot S(S - 3Z) = 0$, then $f^*(K_X)(S - 3Z)^2 \leq 0$, which derives $f^*(K_X) \cdot Z^2 \leq \frac{1}{3}$, i.e. $f^*(K_X) \cdot Z^2 \leq 0$. (A) is also false.

Case (iii). $f^*(K_X) \cdot S(S - Z) = 0$ induces $f^*(K_X) \cdot Z^2 \leq 2$, then (A) becomes $K_X^3 \leq 2$. Thus $K_X^3 = 2$. Actually, in this case, $f^*(K_X) \cdot (S - Z) \sim_{\text{num}} 0$ (as 1-cycle).

Case (iv). $f^*(K_X) \cdot (3S - Z)^2 \leq 0$ induces $f^*(K_X) \cdot Z^2 \leq 9$. And (A) becomes $K^3 \leq 4$. If $K_X^3 = 4$, we see that $f^*(K_X) \cdot (3S - Z) \sim_{\text{num}} 0$ as 1-cycle. Now we set $f^*(M_2) = S + E$, then $Z = f^*(Z_2) + E$. Obviously, we have $f_*(S) = M_2$ and $f_*(Z) = Z_2$. From $f^*(M_2) \cdot f^*(K_X) \cdot (3S - Z) = 0$, we get $3K_X \cdot M_2^2 = K_X \cdot M_2 \cdot Z_2$. Then $4 = 2K_X^2 \cdot M_2 = K_X \cdot M_2^2 + K_X \cdot M_2 \cdot Z_2 = 4K_X \cdot M_2^2$, i.e. $K_X \cdot M_2^2 = 1$. Which derives a contradiction, because $K_X \cdot M_2^2$ is even. Thus $K_X^3 = 2$. $\square$

Proposition 4.2. *Let $X$ be a nonsingular projective 3-fold with nef and big $K_X$. Suppose $K_X^3 > 2$ and* $\dim\phi_1(X) \geq 2$, *then $\delta_1(X) \geq 3$.*

*Proof.* As in the proof of the previous proposition, we first take a modification $f : X' \longrightarrow X$. Set $f^*(K_X) \sim_{\text{lin}} M + Z$, where $M$ is the moving part. A general member $S \in |M|$ is a nonsingular projective surface of general type. Also denote $L := f^*(K_X)|_S$, then $L^2 = \delta_1(X) \geq 2$ according to Proposition 2.1 of [5]. If $L^2 = 2$, then we have

$$2 = f^*(K_X)^2 \cdot S = f^*(K_X) \cdot S^2 + f^*(K_X) \cdot S \cdot Z.$$

We also have

$$\begin{aligned} K_X^3 &= f^*(K_X)^2 \cdot S + f^*(K_X)^2 \cdot Z \\ &= 2 + f^*(K_X) \cdot S \cdot Z + f^*(K_X) \cdot Z^2. \end{aligned} \tag{B}$$

Similarly, $f^*(K_X)\cdot S^2\geq 1$. If $f^*(K_X)\cdot S^2=2$ and $f^*(K_X)\cdot S\cdot Z=0$, then, by the Hodge Index Theorem, $f^*(K_X)\cdot Z^2\leq 0$. Then (B) becomes $K_X^3\leq 2$, which says $K_X^3=2$. If $f^*(K_X)\cdot S^2=f^*(K_X)\cdot S\cdot Z=1$. $f^*(K_X)\cdot S\cdot(S-Z)=0$ induces $f^*(K_X)\cdot Z^2\leq 1$, then (B) becomes $K_X^3\leq 4$. If $K_X^3=4$, then we can see $f^*(K_X)\cdot(S-Z)\sim_{\text{num}}0$. We have $f^*(M_1)\cdot f^*(K_X)\cdot(S-Z)=0$, i.e. $K_X\cdot M_1^2=K_X\cdot M_1\cdot Z_1$. We have $2=K_X^2\cdot M_1=K_X\cdot M_1^2+K_X\cdot M_1\cdot Z_1=2K_X\cdot M_1^2$. Therefore $K_X\cdot M_1^2=1$, which is impossible. Thus $K_X^3=2$. $\square$

Theorem 4.2. *Let $X$ be a nonsingular projective threefold with nef and big $K_X$, then $\phi_5$ is a generically finite morphism of degree $\leq 8$. If $\deg(\phi_5)>2$, then $K_X^3=2$, $\chi(\mathcal{O}_X)=-1$ and $p_g(X)=0,1$.*

*Proof.* According to Theorem 4.1, we only have to study the case when $|2K_X|$ is not composed of pencils. Take a modification $f:X'\longrightarrow X$ according to Hironaka such that $|2f^*(K_X)|$ defines a morphism. Set $2f^*(K_X)\sim_{\text{lin}}M+Z$, where $M$ is the moving part and $Z$ the fixed one. A general member $S\in|M|$ is a nonsingular projective surface of general type by the Bertini Theorem. We have

$$|K_{X'}+2f^*(K_X)+S|\subset|5K_{X'}|.$$

Denote $L:=f^*(K_X)|_S$, using the long exact sequence and Kawamata-Viehweg vanishing, we have

$$|K_{X'}+2f^*(K_X)+S||_S=|K_S+2L|.$$

Note that $h^0(S,\mathcal{O}_S(2L))\geq P(2)-1\geq 3$. We have two cases:

Case 1. $|2L|$ is composed of pencils. Take a birational modification to $S$ if necessary, we can suppose $|2L|$ is free of base points. Denote $2L\sim_{\text{lin}}\sum_{i=1}^a C_i+E$, where $E$ is the fixed part. In general position, $\sum_{i=1}^a C_i$ can be a disjoint union of nonsingular curves in a family. We have $a\geq 2$. Thus $L\sim_{\text{num}}\frac{a}{2}C+E_0$, where $E_0:=\frac{1}{2}E$. If $p_g(S)=0$, then $q(S)=0$ and then we can see by the long exact sequence that $|K_S+2L|$ can distinguish $C_i$'s and that $|K_S+\sum_{i=1}^a C_i||_{C_i}=|K_{C_i}|$, which means $\phi_5$ is at worst a generically finite map of degree 2. If $p_g(S)>0$, it is obvious that $|K_S+2L|$ can distinguish $C_i$'s. For a general curve $C$ which is algebraically equivalent to $C_i$, we consider the $\mathbb{Q}$-divisor $G:=K_S+2L-\frac{1}{2}\sum_{i=3}^a C_i-E_0$. We have $|\lceil G\rceil|\leq|K_S+2L|$. On the other hand, $G-C-K_S$ is nef and big, thus by the Kawamata-Viehweg vanishing we have $|\lceil G\rceil||_C=|K_C+\lceil E_0\rceil|_C|$. Note that $\lceil E_0\rceil|_C$ is effective, therefore $\Phi_{|K_S+2L|}$ is at worst a generically finite map of degree 2, so is $\phi_5$ of $X$.

Case 2. $|2L|$ is not composed of pencils. Similarly, we can suppose $|2L|$ is base point free. If $p_g(S)=0$, we can use a parallel discussion to that of Case 1 to see that $\phi_5$ is at worst a generically finite map of degree 2. If $p_g(S)>0$, then $\Phi_{|K_S+2L|}$ is obviously generically finite. We know that $L^2\geq 2$ from Proposition 2.2 of [2]. If $\Phi_{|K_S+2L|}$ is not birational and $L^2\geq 3$, then according to Corollary 2 of [21], there is a free pencil on $S$ with a general member $C$ such that $C^2=0$ and $L\cdot C=1$. Note that $\dim\Phi_{|2L|}(C)=1$, then $h^0(2L|_C)\geq 2$ and then, by the Clifford theorem, we see that $C$ is a curve of genus 2 and $2L|_C\sim_{\text{lin}}K_C$, finally we can see that $|2L||_C=|K_C|$. Therefore $\Phi_{|K_S+2L|}$ is a generically finite map of degree 2. Therefore $\phi_5$ is generically finite with $\deg(\phi_5)\leq 2$. If $L^2=2$, then $K_X^3=2$ by Proposition 4.1. On the surface $S$, set $2L\sim_{\text{lin}}C_1+E_1$, where $C_1$ is the moving part. We easily get

$$8=(2L)^2\geq C_1^2\geq d(h^0(2L)-2)\geq d(P(2)-3).$$

Therefore we have

$$d \leq \frac{8}{P(2)-3} = \frac{8}{-3\chi(\mathcal{O}_X)-2}.$$

If $d > 2$, then $\chi(\mathcal{O}_X) = -1$. $\square$

For the 4-canonical map of $X$, it is obvious that $\phi_4$ is not birational if $X$ admits a pencil of Horikawa surfaces of general type with $(K^2, p_g) = (1, 2)$. Therefore it is pessimistic to us to obtain an effective sufficient condition for the birationality of $\phi_4$. We have a partial result as follows.

THEOREM 4.3. *Let $X$ be a nonsingular projective 3-fold with nef and big $K_X$. Suppose $K_X^3 > 2$ and* $\dim\phi_1(X) = 3$, *then $\phi_4$ is a birational map onto its image.*

*Proof.* Take a birational modification $f : X' \longrightarrow X$ such that $|f^*(K_X)|$ is base point free. Set $f^*(K_X) \sim_{\text{lin}} S+Z$, where $S$ is the moving part and $Z$ the fixed one. A general member $S$ is a nonsingular projective surface of general type. We have $|K_{X'}+2f^*(K_X)+S| \subset |4K_{X'}|$. Using the vanishing theorem, we have

$$|K_{X'} + 2f^*(K_X) + S||_S = |K_S + 2L|,$$

where $L := f^*(K_X)|_S$ is a nef and big divisor on $S$. By Proposition 4.2, we see that $L^2 \geq 3$ under the condition $K_X^3 > 2$. If $\Phi_{|K_S+2L|}$ is not birational, then, by Corollary 2 of [21], there is a free pencil with a general member $C$ such that $C^2 = 0$ and $L \cdot C = 1$. Because $\dim\Phi_{|L|}(S) = 2$, $h^0(C, \mathcal{O}_C(L|_C)) \geq 2$. Therefore, by the Clifford theorem, we see that $\deg(L|_C) \geq 2h^0(L|_C) - 2 \geq 2$. This is a contradiction. Therefore $\Phi_{|K_S+2L|}$ is birational and so is $\phi_4$. $\square$

EXAMPLE 4.1. We give an example which shows that $\phi_4$ is not birational when $K_X^3 = 2$ and $\dim\phi_1(X) = 3$. On $\mathbb{P}^3(\mathbb{C})$, take a smooth hypersurface $S$ of degree 10, $S \sim_{\text{lin}} 10H$. Let $X$ be a double cover of $\mathbb{P}^3$ with branch locus along $S$, then $X$ is a nonsingular canonical model, $K_X^3 = 2$ and $p_g(X) = 4$ and $\phi_1$ is a finite morphism onto $\mathbb{P}^3$ of degree 2. One can easily check that $\phi_4$ is also a finite morphism of degree 2.

THEOREM 4.4. *Let $X$ be a nonsingular projective 3-fold with nef and big $K_X$, then $\phi_4$ is generically finite when $p_g(X) \geq 3$ or when $K_X^3 > 2$ or when $\chi(\mathcal{O}_X) \neq -1$.*

*Proof.* Part I: $p_g(X) \geq 3$.

Firstly, we make a modification $f : X' \longrightarrow X$ such that $|f^*(K_X)|$ is free from base points and that $f^*(K_X)$ has support with only normal crossings. Set $f^*(K_X) \sim_{\text{lin}} M + Z$, where $M$ is the moving part and $Z$ the fixed one.

If $\dim\phi_1(X) = 2$, then a general member $S \in |M|$ is a nonsingular projective surface of general type. We have

$$|K_{X'} + 2f^*(K_X) + S| \subset |4K_{X'}|.$$

Using the vanishing theorem, we have $|K_{X'} + 2f^*(K_X) + S||_S = |K_S + 2L|$, where $L := f^*(K_X)|_S$ is nef and big effective divisor on $S$. We have $h^0(S, L) \geq 2$. Noting that $p_g(S) > 0$ in this case, if $|L|$ is not composed of pencils, then neither is $|K_S + 2L|$. If $|L|$ is composed of pencils, taking a modification if possible, we can suppose $|L|$ is free from base points. Set

$L \sim_{\text{lin}} \sum C_i + Z_0$, we can see $|K_S + L + \sum C_i||_{C_i} = |K_{C_i} + D|$, where $D := L|_{C_i}$ is effective. We easily see that $\Phi_{|K_S+2L|}$ is at worst generically finite of degree $\leq 2$ and so is $\phi_4$.

If $\dim\phi_1(X) = 1$, then $M \sim_{\text{num}} aF$, where $F$ is a nonsingular projective surface of general type. $M_1 \sim_{\text{num}} aF_0$, where $F_0 = f_*(F)$ is irreducible on $X$. If $K_X \cdot F_0^2 = 0$, then, by Lemma 2.3 of [5], we have $\mathcal{O}_F(f^*(K_X)|_F) \cong \mathcal{O}_F(\pi^*(K_0))$, where $\pi$ is the contraction onto the minimal model and $K_0$ is the canonical divisor of the minimal model of $F$. We see that $|K_{X'} + 2f^*(K_X) + M||_F = |K_F + 2\pi^*(K_0)|$, the latter system defines a generically finite map and so does $\phi_4$. If $K_X \cdot F_0^2 > 0$, because $p_g(X) \geq 3$, we have $a \geq 2$. Consider the $\mathbb{Q}$-divisor $G := 3f^*(K_X) - \frac{1}{a}Z$, $G - F$ is nef and big, therefore by the Kawamata-Viehweg vanishing, we get

$$|K_{X'} + \lceil G \rceil||_F = |K_F + 2L + \lceil \frac{a-1}{a} Z \rceil|_F|.$$

By Lemma 2.1, we have

$$|K_F + 2L + \lceil \frac{a-1}{a} L \rceil| \subset |K_F + 2L + \lceil \frac{a-1}{a} Z \rceil|_F|.$$

We want to show that the former system defines a generically finite map. If $\dim\phi_2(X) = 1$, then, by Claim 9.1 of [17], $\mathcal{O}_F(f^*(K_X)|_F) \cong \mathcal{O}_F(\pi^*(K_0))$. Thus $\Phi_{|K_F+2L|} \approx \Phi_{|3K_F|}$ is generically finite. So we may suppose $\dim\phi_2(X) \geq 2$, then $\phi_2(F) \geq 1$, which says $h^0(2L) \geq 2$. Using Theorem 2.1 on the surface $F$, we see that $|K_F + 2L + \lceil \frac{a-1}{a} L \rceil|$ defines a generically finite map. Thus $\phi_4$ is generically finite.

Part II: $K_X^3 > 2$ or $\chi(\mathcal{O}_X) \neq -1$.

We study $\phi_4$ according to the behavior of $\phi_2$. Of course, first we make a modification $f : X' \longrightarrow X$ such that $|2f^*(K_X)|$ is free from base points and that $2f^*(K_X)$ has support with only normal crossings. Set $2f^*(K_X) \sim_{\text{lin}} \overline{M_2} + \overline{Z_2}$, where $\overline{M_2}$ is the moving part and $\overline{Z_2}$ the fixed one.

If $\dim\phi_2(X) = 1$, then $\overline{M_2} \sim_{\text{num}} a_2 F$, where $F$ is a nonsingular projective surface of general type. As we have seen in the above, $\mathcal{O}_F(f^*(K_X)|_F \cong \mathcal{O}_F(\pi^*(K_0))$. We have $|K_{X'} + f^*(K_X) + \overline{M_2}||_F = |K_F + \pi^*(K_0)|$. From Theorem 3.1 of [5], we know that $F$ is not a surface with $p_g = q = 0$. Thus $|K_F + \pi^*(K_0)|$ defines a generically finite map and so does $\phi_4$.

If $\dim\phi_2(X) \geq 2$, then a general member $S \in |\overline{M_2}|$ is a nonsingular projective surface of general type. We have $|K_{X'} + f^*(K_X) + S||_S = |K_S + L|$, where $L := f^*(K_X)|_S$. Note that $K_S \geq L$, then $K_S + L \geq 2L$. Under our assumption, we have $P(2) \geq 5$. Thus $h^0(2L) \geq 4$. We may suppose $|2L|$ is free from base points. If $|2L|$ is not composed of pencils, then nor is $|K_S + L|$. Otherwise we can set $2L \sim_{\text{lin}} \sum_{i=1}^{b} C_i + E_1$, where $b \geq 3$ and $E_1$ is the fixed part. We denote the $C_i$ by $C$ generally. Because $L - C - \frac{1}{b}E_1$ is nef and big, therefore

$$|K_S + \lceil L - \frac{1}{b}E_1 \rceil||_C = |K_C + \lceil \frac{b-2}{2b} E_1 \rceil|_C|.$$

The latter system obviously defines a generically finite map. Thus $\phi_4$ is also generically finite. $\square$

THEOREM 4.5. *Let $X$ be a nonsingular projective 3-fold with nef and big $K_X$, then $\phi_3$ is generically finite when $p_g(X) \geq 39$.*

*Proof.* Firstly, we make a modification $f: X' \longrightarrow X$ such that $|f^*(K_X)$ is free from base points and that $f^*(K_X)$ has support with only normal crossings. Set $f^*(K_X) \sim_{\rm lin} M + Z$, where $M$ is the moving part and $Z$ the fixed one.

If $\dim\phi_1(X) \geq 2$, then a general member $S \in |M|$ is a nonsingular projective surface of general type. We have $|K_{X'} + f^*(K_X) + S||_S = |K_S + L|$, where $L := f^*(K_X)|_S$. When $p_g(X) \geq 4$, $h^0(S, L) \geq 3$. Noting that $p_g(S) > 0$, if $|L|$ is not composed of pencils, then nor is $|K_S + L|$. So we may suppose $|L|$ is composed of pencils and be free from base points. Set $L \sim_{\rm lin} \sum_{i=1}^{a} C_i + E_0$, where we have $a \geq 2$. If $p_g(S) = 0$, then $q(S) = 0$ and $|K_S + \sum_{i=1}^{a} C_i||_{C_i} = |K_{C_i}|$. The theorem is proved. If $p_g(S) > 0$, then $|K_S + L|$ can distinguish the $C_i$ generically. On the other hand, $L - C - \frac{1}{a}E_0$ is nef and big, we obtain by the Kawamata-Viehweg vanishing that

$$|K_S + \lceil L - \frac{1}{a}E_0\rceil||_C = |K_C + \lceil\frac{a-1}{a}E_0\rceil|_C|.$$

The latter system defines a generically finite map and so does $\phi_3$.

If $\dim\phi_1(X) = 1$, we keep the same notations as in Part I of Theorem 4.4. If $K_X \cdot F_0^2 = 0$, then we only have to prove that $|K_F + \pi^*(K_0)|$ defines a generically finite map, which is obvious since $p_g(F) > 0$. If $K_X \cdot F_0^2 > 0$, in order to prove the theorem, we have to show the generic finiteness of $\Phi_{|K_F+L|}$, where $L := f^*(K_X)|_F$ is effective. By Theorem 2 of [3], we see that $q(F) \geq 3$ when $p_g(X) \geq 39$. Then $\Phi_{|K_F|}$ is generically finite according to Xiao [24]. Therefore under the assumption of the theorem, we can obtain the generic finiteness of $\phi_3$. $\square$

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